Observing mathematical practices as a key to mining our sources and conducting conceptual history.

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OBSERVING MATHEMATICAL PRACTICES AS A KEY TO MINING OUR SOURCES AND CONDUCTING CONCEPTUAL HISTORY. DIVISION IN ANCIENT CHINA AS A CASE STUDY

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The present chapter can be read as a reflection inspired by the working conditions of the historian of ancient mathematics. By contrast to colleagues working on early modern or modern time periods, the historian of the ancient world has in general very few documents on which to rely. In addition, these sources were often produced decades or even centuries apart. Owing to complex historical processes usually difficult to investigate, these sources happen to have been handed down, unlike many other documents of the past dealing with similar topics.[2] The historian has to work with them, despite the fact that they can at best be put in a rarefied historical context, if they can be put in a context at all. These conditions impose that historians devote the greatest attention to methods allowing them to derive as much information as possible from these scarce documentary resources. How to make sources speak is a question of general interest for historians. It is a vital issue for ancient history.

The difficulty is compounded when we want to address questions that our sources evoke only tangentially or indirectly —we shall meet with specific examples below. Similar issues have been debated for decades in the field of general history. This was, in particular, the case when historians like Carlo Ginzburg attempted to derive insight about actors who left no written documents, on the basis of documents that had been produced by others.[3] Carlo Ginzburg has addressed this issue theoretically, by

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[2] Florence Bretelle-Establet, ed., *Looking at it from Asia: the processes that shaped the sources of history of science*, vol. 265, Boston Studies in the Philosophy of Science (2010) is an attempt to address this issue.

[3] Carlo Ginzburg, *Threads and Traces. True False Fictive* (Berkeley/Los Angeles/London, 2012):3-4 summarizes the gist of method, referring the reader to the theoretical insights of Marc Bloch, *Apologie pour l'histoire, ou Métier d'historien* (Paris, 1949). In this posthumous book, Marc Bloch offers thoughts about the key part played by traces in the

discussing the notion of "clue," or "trace," and sketching the history of the "evidential paradigm."[4]

In my view, the implications of these debates for conceptual history, let alone conceptual history of mathematics in the ancient world, have not, as far as I know, been addressed. How can we account for concepts and bodies of knowledge on which our sources only leave clues? The present chapter is an attempt to deal with this question. The thesis for which I shall argue is that restoring mathematical practices in relation to which our sources were produced yields key resources to inquire into issues of conceptual history on which our sources do not dwell.[5] The description of practices, I claim, allows us to capture elements of actors' knowledge to which our sources testify but that would otherwise remain out of reach. In other words, the description of practices helps interpret clues. Moreover, it allows us to perceive changes in actors' bodies of knowledge and thus to grasp questions to which actors devoted some attention, even though we have no direct evidence for this fact. In brief, the chapter aims at demonstrating *how* the description of practices can be essential to carry out conceptual history, through the resources it provides to make sense of clues.

To flesh out these issues and illustrate the kind of actual evidence by means of which one can support the claims made above, I shall unfold the argument in the context of a case study. This case study, devoted to part of the mathematical work done on arithmetical operations, and more specifically on division, in ancient China, is introduced in the first part. In the second part, I shall outline aspects of a state of knowledge on this topic, as it can be captured on the basis of a book composed in China in the first century CE and related evidence. There, the argument will piece together evidence on this state of knowledge and information we have acquired on mathematical practices at the time. The third part will contrast the results obtained with features of the state of knowledge on the same topic to which other writings, produced in the preceding centuries in China, testify. The comparison highlights that the state of knowledge evidenced in the first century must have been the result of a work for which we so far have no other written evidence. The conclusion will draw some general remarks from this specific case study.

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approach of the historian at his time. In particular, he writes (p. 25): "(…) jusque dans les témoignages les plus résolument volontaires, ce que le texte nous dit expressément a cessé aujourd'hui d'être l'objet préféré de notre attention. Nous nous attachons ordinairement avec bien plus d'ardeur à ce qu'il nous laisse entendre, sans avoir souhaité le dire. (…) Dans notre inévitable subordination envers le passé nous nous sommes donc affranchis du moins en ceci que, condamnés toujours à le connaître exclusivement par ses traces, nous parvenons toutefois à en savoir sur lui beaucoup plus long qu'il n'avait lui -même cru bon de nous en faire connaître. C'est, à bien le prendre, une grande revanche de l'intelligence sur le donné."

[4] See Carlo Ginzburg, "Traces. Racines d'un paradigme indiciaire", in *Mythes, emblèmes, traces. Morphologie et histoire*, ed. Carlo Ginzburg (Paris, 1989). Which traces a text leaves and what historians can draw from them appear to me to remain questions worth addressing theoretically. I leave this discussion for another publication.

[5] How we can rely on our sources to restore practices is another topic, which I have addressed in several other publications. I shall mention some of these publications below, when I need to make use of their results for my argument.

I. The trouble with researching arithmetical operations in ancient China—A case study

In various parts of the planet in the ancient world, arithmetical operations have been a topic of theoretical reflection. However, historians have not explored these reflections systematically. Probably, the main problem that confronted historians was the lack of direct evidence. Spotting traces of these reflections and interpreting them are two delicate undertakings. They will be precisely the topic addressed in this chapter. Let us explain why these enterprises are tricky in the case of ancient China.

Mathematical documents that have survived from early imperial China, that is, documents produced between the third century BCE and the first century CE, show that "procedures" —in present-day terminology, "algorithms," in Chinese "*shu* 術"— were at the focus of attention for the practitioners of mathematics. In relation to this fact, we can perceive that operations were key objects of study.

On the one hand, arithmetical operations were the building blocks of procedures. One can illustrate this assertion with, e.g., the procedure computing the volume of a half-parallelepiped. It consists of a sequence of two multiplications followed by a division by 2 and it aims at carrying out a task formulated by a mathematical problem. This coarse description shows that building blocks —in this case, multiplication and division by 2— were shaped to fulfill this function. These building blocks are what we call operations. When we meet, in procedures, with operations like "dividing in return *baochu* 報除," we understand that we cannot take the shaping of building blocks for granted and that it calls for closer examination than it is usually given.

On the other hand, operations were also executed by means of procedures. One can think, for instance, of square root extraction, as an operation for the execution of which algorithms are provided in some ancient Chinese mathematical sources. However, for most operations, at the time period considered, we know procedures executing them for their most part only indirectly. These examples suffice to show that the relationship between operations and procedures is far from obvious.

In fact, we can perceive *how* operations were objects of study, and the outcome of this inquiry, by means of sources that are texts mainly composed of mathematical problems, algorithms solving them and numerical tables. These documents put operations into play: names for operations occur in texts of procedures. Moreover, these sources contain evidence on ways of working with operations outside the text. For instance, they give fragmented evidence on the computing instruments with which operations were executed or on diagrams associated to them. However, they bear witness to ideas on operations only indirectly. In other words, these are sources that probably referred to actions, such as the execution of operations. They adhere to a practice of mathematics on which they provide clues. But they are by no means treatises about operations. In fact, no discursive treatment of operations survived from early imperial China, if any ever existed.

The state of the surviving documentation thus raises a problem for the historian who wants to inquire into the theoretical work done on operations. With respect to the subject matter, by means of which methods can we observe the knowledge shaped in ancient China about operations? How can we capture ideas that were formed about operations or the goals that were assigned to inquiries on this topic? In addition, with respect to the practice, how can we restore the ways of working with operations that were designed? More generally, how can we perceive a history of the bodies of knowledge formed about operations, by relying on sources that document all these aspects indirectly?

The issue is tricky. However, we cannot leave these questions unattended, since something essential is at stake. Clearly significant efforts have been devoted to operations and their outcomes represent an important, even an essential, part of the theoretical work on mathematics carried out in ancient China. As was already mentioned above, I shall take for granted the results of the articles in which I addressed the question of restoring practices to focus here on *how* one can bring to light the conceptual work done on operations.

II. Weaving clues. Aspects of a state of knowledge on division and related operations

The main document on which I shall rely in this section is the book that was probably composed in the first century CE and that became a Classic soon afterwards, that is, *The Nine Chapters on mathematical procedures* (*Jiuzhang suanshu* 九章算術). Even though in what follows, I abbreviate the title into *The Nine Chapters*, it is interesting to note that the original title reveals the importance of procedures in the eyes of those who composed the book. Like all Classics, this one was the object of commentaries, two of which were selected by the written tradition to be handed down with *The Nine Chapters*. These are the commentary that Liu Hui completed in 263 and the one supervised by Li Chunfeng and presented to the throne in 656. For the purpose of our argument, we will need below to refer to these texts.

The objective of this section is to show that *The Nine Chapters* bears witness to actors possessing at the time a quite complex and elaborate knowledge about a set of operations, in which division played the central part.[6] Yet, as we shall see, the operation of division is one on which we have only very little evidence. It thus requires a complex argument to establish this fact. Let us observe step by step how one can weave clues to reach this conclusion.

To begin with, the texts of procedures in *The Nine Chapters* cast light on features of the computing instrument used at the time in relation to the book. First, they often refer to the action of "placing 置 *zhi*" values, which testifies to the existence, by the text, of an "instrument," probably a mere surface, on which a representation of values was placed and with which one computed. Second, *The Nine Chapters* also indicates that on that base "counting rods" were used to represent numbers. In fact, the operations prescribed by its procedures were executed on numbers represented with counting rods

[6] When I speak of division in this chapter, it designates only division between integers or between integral amounts of measuring units. In Karine Chemla, "Changing mathematical cultures, conceptual history and the circulation of knowledge. A case study based on mathematical sources from ancient China", in *Cultures without culturalism*, eds. K. Chemla and Evelyn Fox-Keller (To appear), I have dealt with the same pieces of evidence from *The Nine Chapters*. There, the focus was to highlight a correlation between the concepts and the mathematical practice in relation to which these concepts were shaped. Moreover, this conclusion served as a basis to establish, by a forward-looking argument, that concepts are not determined by the scholarly culture to which they adhere. The purpose here is different, since I intend to bright to light actors' knowledge with respect to operations and to focus on the method with which this can be done. In the following section, I shall rely on the conclusions obtained in this section to make a backward-looking argument.

on this surface, and apparently not on any writing material. Which was the number system according to which numbers were written down with rods? *The Nine Chapters* states nothing on that matter, but we shall see below that we can get indirect evidence for that. Third, the procedures of *The Nine Chapters* further make clear that in the context of a computation, values could be put in different positions (above, below, in the middle, etc).[7] We have here instances of clues that a writing indirectly gives on a feature of mathematical practice, here the practice of computing. These clues bear witness to an object which is outside the text, and on which the text makes no discursive development.[8] Contrary to what most historians have claimed so far, the three features mentioned above do not suffice to identify the computing instrument and claim that the instrument has been the same from the third century BCE up to the fourteenth century. If this conclusion holds true, this case illustrates altogether how we can find clues in writings, which provide indirect evidence on mathematical practice, and how delicate an operation the interpretation of these clues represents.

Indeed, all the manuscripts discovered recently, to which I return in the next section, also contain these three clues. However, I have come to the conclusion that they do not refer to the same instrument as *The Nine Chapters*, unless they do not refer to the same way of using the instrument.[9] Key differences between the practice of computing to which *The Nine Chapters* bears witness and that to which the manuscripts testify can be captured if we observe the basic operations on the representation of values with rods to which their texts of procedures refer. For instance, procedures in *The Nine Chapters* regularly make use of the fact that rods, or representations of numbers, can be *moved* forward or backward on the surface. The utilization of these basic moves implies that the number system used in relation to *The Nine Chapters* must have had features that made these operations meaningful. By contrast, as far as I can tell, there is no reference to any basic operation of that kind in the manuscripts. The contrast is essential to keep in mind. Indeed, these additional clues, derived from how operations echo the material

[7] In Karine Chemla, "Positions et changements en mathématiques à partir de textes chinois des dynasties Han à Song- Yuan. Quelques remarques", in *Disposer pour dire, placer pour penser, situer pour agir. Pratiques de la position en Chine*, eds. Karine Chemla and Michael Lackner, Extrême-Orient, Extrême-Occident (Saint-Denis, 1996), I have described the evidence we have on this system of computation.

[8] Rods have also been found in archeological excavations of tombs sealed in Qin and Han dynasty. However, this evidence is difficult to interpret, since we cannot establish with certainty in the context of which activities these rods were used. Reference in mathematical writings is more reliable evidence in this case.

[9] I shall come back to this issue in another publication. Among the authors who have claimed that the instruments were the same, see Christopher Cullen, *The Suan shu shu 筭數書 'Writings on reckoning': A Translation of a Chinese mathematical collection of the second century BC, with explanatory commentary*, ed. Christopher Cullen, vol. 1, *Needham Research Institute Working Papers* (Cambridge, 2004): 24; Joseph W. Dauben, "算數書. Suan Shu Shu (A Book on Numbers and Computations). English Translation with Commentary", *Archive for history of exact sciences* 62 (2008): 96. Before new mathematical manuscripts had been excavated from tombs, archeologists had discovered rods. Moreover, some early written documents mention ways of computing. The assumption was prevalent that the rod system kept unchanged. See, e.g., Lam Lay Yong, "A Chinese Genesis: Rewriting the History of Our Numeral System", *Archive for history of exact sciences* 38 (1988).

properties of the number systems to which they are applied, seem to indicate that we should be careful not to assume that all these writings were composed by reference to the same number system, even though in all cases the number system was written down with counting rods on the surface.

For the time being, let us limit our discussion to *The Nine Chapters*. We shall meet below with further evidence that the book provides about the practice with the computing instrument. Indeed, since operations are executed on the basis of a computing instrument and a given number system, it is not surprising that their execution betrays features of that on which they operate. Conversely, the knowledge we can derive from texts of procedures about the instrument and number system helps us interpret these texts. We shall come back to this interplay in the conclusion of the chapter.

II.1 *Division in* The Nine Chapters

Which evidence does *The Nine Chapters* provide with respect to division?

To begin with, the texts of procedures in *The Nine Chapters* abound in prescriptions of division. Several terms are used for that purpose. It will prove useful below to say a few words about them. On the one hand, division can be prescribed by a verb "divide 除 *chu*." On the other hand, the operation is also designated by one among a set of related expressions : "(quantity) then one 而一 *er yi*," which indicates that the dividend is divided by the "quantity," each part of the dividend equal to the "quantity" becoming 1;[10] "like (quantity) then one 如…而一 *ru* (quantity) *er yi*"; and finally "when the dividend is like the divisor, then one" or "then it yields one," or else "then one (name of a measuring unit)," "then it yields one (name of a measuring unit)"(in Chinese: *shi ru fa er yi* 實如法而一， *shi ru fa de yi* 實如法得一， *shi ru fa er yi* (name of a measuring unit) 實如法而一(name of a measuring unit), *shi ru fa de yi* (name of a measuring unit) 實如法得一(name of a measuring unit)).[11] In *The Nine Chapters*, only for the operation of division do we find prescriptions by other means but verbs.[12]

Seen from the way in which it is prescribed in the text of procedures, division is different from all the other arithmetic operations in another respect. It is the only operation, for which technical terms were introduced to designate the operands,

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[10] Incidentally, this is the expression prescribing the division that is used in the text of the procedure computing the volume of the half-parallelepiped, mentioned above.

[11] For any remark on the terminology, unless otherwise stated, I refer the reader to the glossary I published in Karine Chemla and Guo Shuchun, *Les neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires* (Paris, 2004). In the glossary, I discuss the terms used to prescribe operations and I give information on the syntax of the sentences in which these terms are employed. Moreover, I refer to evidence supporting my conclusions. However, for the last set of expressions, I shall come back below to their original meaning on the basis of evidence provided in the next section. Li Jimin 李繼閔, *Jiuzhang suanshu daodu yu yizhu 九章算術導讀與譯註 (Guidebook and annotated translation of The Nine Chapters on Mathematical Procedures)* (Xi'an, 1998): 144-148 discusses the terminology of division in *The Nine Chapters.*

[12] We shall see below that division has cognate operations and that they are *only* prescribed by means of the verb *chu*, to which qualifications are added. In what follows, when I speak of "division" in the context of *The Nine Chapters*, it usually means division and its cognate operations.

"dividend *shi* 實", "divisor *fa* 法". This feature will also prove important below. When, e.g., a multiplication is prescribed, its operands are indicated by means of values or expressions referring to the magnitudes whose values are to be multiplied. By contrast, the prescription of a division often describes how the dividend and divisor are obtained, before inserting one of the technical expressions referring to the operation, which is to be executed on the basis of the operands.[13] Let us emphasize that in this case, technical terms bear witness to an understanding of an operation as, on the one hand, having operands and, on the other hand, being associated to a procedure that will be executed on these operands.

The Nine Chapters does not contain any text of procedure allowing the reader to execute a division. The same holds true for addition, subtraction, and multiplication. Only in later texts, that is, e.g., in the *Mathematical Classic by Master Sun* (*Sunzi suanjing* 孫子算經), completed in 400 CE, do we have explicit algorithms for multiplication and division (the latter is then called *chu*).[14] Is this procedure for division the same one as the one meant by the authors of *The Nine Chapters*? We shall show below how we can find in *The Nine Chapters* indirect evidence, based on our knowledge of mathematical practice at the time, to establish that this was the case. In addition, the *Mathematical Classic by Master Sun* makes clear how the algorithm is deployed on the surface that served as a computing instrument and how numbers are represented with rods. These indications show that the number system used to write down numbers with rods was a place-value decimal system. The same question can be raised: can we establish that a similar system was used in the environment that produced *The Nine Chapters*? Here again, the same type of indirect evidence will allow us below to reply positively.

II. 2 *Root extraction in* The Nine Chapters

The examination of root extraction in *The Nine Chapters* will play a pivotal role in my argumentation. I shall not repeat what I have written elsewhere on the matter. My purpose here will be only to bring to light the kind of knowledge about operations to which *The Nine Chapters* testifies and to show how historians need to rely on the

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[13] This description is not entirely correct. Sometimes, for instance, only the dividend is explicitly designated, and the expression prescribing division has a complement referring to the value or the magnitude taken as divisor. A coarse description is enough for our purpose here.

[14] Qian Baocong 錢寶琮, *Suanjing shishu 算經十書 (Qian Baocong jiaodian 錢寶琮校點) (Critical punctuated edition of The Ten Classics of Mathematics)*, 2 vols. (Beijing 北京, 1963), vol. 2: 282-283, provides a critical edition. Qian Baocong argues that the book was composed around 400 CE, but stresses that the received version displays hints of later, Tang, changes. Lam Lay Yong and Ang Tian Se, *Fleeting footsteps: Tracing the conception of arithmetic and algebra in ancient China. Revised edition*, Rev. ed. (River Edge, NJ, 2004): 194-195 provides a coarse translation. The algorithm is also explained in Chemla and Guo Shuchun, *Les neuf chapitres*: 16-19. I discuss the key features and terms of these texts in Karine Chemla, "Positions et changements en mathématiques à partir de textes chinois des dynasties Han à Song- Yuan. Quelques remarques", in *Disposer pour dire, placer pour penser, situer pour agir. Pratiques de la position en Chine*, eds. Karine Chemla and Michael Lackner, Extrême-Orient, Extrême-Occident (Saint-Denis, 1996).

description of mathematical practices to reach these conclusions. The type of clues that the Classic provides on root extraction is different from those we found for division.

In texts of algorithms, *The Nine Chapters* refers to operations of square root extraction and also cube root extraction, though the latter occurs less frequently. Two families of expressions are used to prescribe them. Sometimes, root extraction, no matter whether it is square root or cube root extraction, is referred to by the verb "*kai* 開," for which I suggest the translation "extract the root of." This term is used only in contexts in which algorithms to execute root extraction are under discussion. This fact explains why, even though each time a specific kind of extraction is meant, the verb used can be non-specific. The context makes clear the actual meaning of the term.[15] Much more often, the two operations are prescribed by expressions of the utmost importance for us. I translate them in such a way as making explicit the structure of the terminology in classical Chinese: "divide this by extraction of the square root *kai fang chu zhi* 開方除之" (literally "divide (*chu*) this (*zhi*) by opening (*kai*) the square *(fang)*"), and "divide this by extraction of the cube root *kai lifang chu zhi* 開立方除之" (literally "divide (*chu*) this (*zhi*) by opening (*kai*) the cube (*lifang*).") Let us comment on some features of these terms.

First, the anaphora "this *zhi*" designates the value to which the operations are to be applied. Like in the case of division, and in contrast to the other arithmetical operations, a technical term is introduced to name the operand of root extractions. Interestingly enough, in both cases, this term is the one used in the context of division to refer to the "dividend," namely, *shi* 實. The fact and the term both connect root extractions and division. As a side remark, note that extractions thus appear, at this stage, as operations that bear on a single operand.

Second, clearly, in contrast to the former terminology discussed ("*kai* 開 extract the root"), the latter expressions used to prescribe root extractions manifest a link between the execution of these operations and the operation of division. Note a detail that will prove useful later: for this purpose, among the many expressions that could have been used to designate division in this context, it is the prescription by the verb *chu* that is chosen. Each of the types of root extraction is prescribed by a formulation that qualifies the term *chu*, thereby stating which kind of division is meant. The qualifications introduced, that is, "by extraction of the square root" and "by extraction of the cube root," have parallel structures. As a whole, the terminology shows that division is the most fundamental operation, and that the two types of root extraction derive in similar ways from it. An examination of the ways of shaping terminologies in the context of *The Nine Chapters* seems to support the hypothesis that the choice of terms is a way of

[15] As I suggested in the glossary published in Chemla and Guo Shuchun, *Les Neuf Chapitres*: 945, the term *kai* is probably a synonym of the term *qi* 啟 "to open, to detach," which it replaced at some point in time, probably in the mid-first century BCE. However, one can note a shift, which in the overall transformation described in this chapter is interesting: *qi* is a verb whose complement is the object to be detached (as in "detach the side of the square"), whereas *kai* takes as its complement the area, the side of which is sought for ("open the area of the square.") As is discussed below, it is precisely the latter area that, in *The Nine Chapters*, is the operand of the operation designated as "dividend," *shi* 實.

stating this fact about the relationship between the three operations.[16] We meet here with a first example where knowledge about a practice can be used to gain insight in actors' knowledge, in this case, about operations.

The terms prescribing root extractions and the term used to designate their operand both indicate that actors established a relationship between the three operations under discussion. This conclusion is confirmed, if we now turn to the texts of the algorithms that *The Nine Chapters* contains and that explain how to carry out the operations of root extraction.[17] In fact, these texts allow us to gain a much deeper insight into how actors understood at the time the relations between the operations.

The texts describing algorithms for the execution of, respectively, square root extraction and cube root extraction are written in close parallel with each other. They correspond to each other sentence by sentence and make use of the same terms. What the texts manifest echoes what the expressions prescribing the operations stated. This way of writing texts for algorithms that display, in the working of the algorithms, a relationship between the operations executed is part of a practice that can be evidenced up to at least the 13th century.[18] The fact that the practice presents a certain stability does not mean, however, that the meanings expressed in this way are the same. In fact, one can perceive differences in the relationship between the same operations thereby expressed, and consequently differences in how the relationship is understood.

The identification of this practice is essential in several ways for us to draw information from *The Nine Chapters*. First, it enables us to assert that the authors of the algorithms shape and state a knowledge about the relationship between the operations of root extraction by means of writing down texts prescribing how the operations can be executed. We can thus not only analyze this relationship but also examine its reformulation in later texts.

Second, these texts for root extractions in *The Nine Chapters* are clearly written by reference to an algorithm for division. The terminology used includes: "dividend," "quotient," "divisor," as well as several technical terms for operations similar to what we find in the text of the *Mathematical Classic by Master Sun* mentioned above. This reference to division again echoes what the mode of prescribing the operations, which we examined above, expressed. We can deduce, from our knowledge of the practice of

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[16] See chapter D in Chemla and Guo Shuchun, *Les neuf chapitres*, especially pp. 99-116, which deals with this question.

[17] Li Jimin 李繼閔, *Dongfang shuxue dianji Jiuzhang suanshu ji qi Liu Hui zhu yanjiu 東方數學典籍——《九章算術》及其劉徽注研究 (Research on the Oriental mathematical Classic The Nine Chapters on Mathematical Procedures and on its Commentary by Liu Hui)* (Xi'an, 1990): 91-105. In Chemla and Guo Shuchun, *Les neuf chapitres*: 19-20, 322-335, 362-368, 370-377 and related footnotes, I give all necessary details on these algorithms and the interpretation of the texts. I refer the reader to this publication for a bibliography on the topic, limiting myself here to the issues specific to this chapter.

[18] K. Chemla, "Qu'apporte la prise en compte du parallélisme dans l'étude de textes mathématiques chinois ? Du travail de l'historien à l'histoire du travail", *Extrême-Orient, Extrême-Occident* 11 (1989), describes this practice. It also contains a translation of the texts for root extraction included in *The Nine Chapters*. An English translation can be found in the appendices of K. Chemla, "Similarities between Chinese and Arabic mathematical writings. I . Root extraction", *Arabic Sciences and Philosophy. A Historical Journal* 4 (1994b). I refer the reader to these translations to follow the argument more easily.

writing down texts for algorithms, that these texts also state, in the same way, a knowledge about the relationship between root extractions and division. Note that later texts expressed in the same way relationships between the same operations that were partly similar and partly different from a theoretical viewpoint.[19] This conclusion confirms that this practice of stating knowledge yielded texts that were read in this way and served as a base for further reworking.

Third, if we consider what the previous conclusion implies with respect to *The Nine Chapters*, we see that that the texts describing algorithms to execute root extraction in the Classic *indirectly* provide us information about the procedure for division, to which they refer. Although no algorithm to carry out division is inserted in *The Nine Chapters*, we can nevertheless derive, from our knowledge of a mathematical practice, some understanding of the algorithm used at the time. In fact, the algorithm for division, by reference to which these texts for root extraction are written, appears to be the same as the one described in the *Mathematical Classic by Master Sun*. We now see the various types of insight that the description of practices yields with respect to actors' knowledge, in a case in which the only documents we have are texts for algorithms. We also begin to perceive how much poorer our information on their knowledge of operations would be, if we only granted actors the knowledge of certain algorithms, that is, merely what appears at the surface of the sources. But, as we shall see, there is more.

Let us go back to a remark made above, about the operand of root extraction. As I emphasized, root extractions are operations that have a single operand, a "dividend." And yet I also pointed out that the texts of algorithms given to execute these operations put into play a "divisor." How do these two facts hold together? The point is that in shaping root extractions as divisions in the process of their execution, the algorithms constitute, progressively along the flow of computation, a "divisor," more precisely a "fixed divisor," which plays the part of the related operand in the process of execution of a division. In the context of root extraction, the "divisor" is thus not an operand. It is a technical component of the process of execution of root extraction, which appears only when one describes the execution. This choice of terminology brings to light an interesting fact about operations. Technical terms associated to them are not limited to the operands, or the name of the operation itself. They can also disclose actors' identification of specific entities that are needed for the process of computation executing the operation. These elements manifest another dimension of the work actors have carried out in the analysis of an operation. The importance of the fact will become clearer below.

The texts for the algorithms executing root extraction do not "describe" how the execution is carried out on the surface that was the computing instrument at the time. They provide clues on the actual computation only incidentally. What can be deduced from these clues? Clearly again, the texts betray the fact that the sequences of events by means of which the square root and the cube root extractions were carried out presented strong correlations. Note that these correlations established connections between the dynamical processes of execution on the surface. For instance, the texts reveal that the executions of square root and cube root extractions make use of the same ways of moving the entities placed in corresponding positions on the surface. Likewise, the executions appear to rely on a similar opposition between key positions and

[19] Karine Chemla, "Similarities between Chinese and Arabic mathematical writings. I . Root extraction", *Arabic Sciences and Philosophy. A Historical Journal* 4 (1994b), is devoted to this question.

auxiliary positions. The description of the practice with the computing instrument to which *The Nine Chapters* and the *Mathematical Classic by Master Sun*, among other sources bear witness, provides a backdrop against which we can interpret these clues. I have shown elsewhere that establishing material relationships in this way between the layouts to execute operations and the sequences of events occurring in each of the positions of the layout on the surface was another way actors shaped to express, and work on, the mathematical relationships between operations.[20] We thus see yet another practice, coupled to the ones examined above, by means of which operations could be a topic of inquiry. Again in this case, the practice manifests a strong stability. It can be evidenced as late as the 13th century and yet the meanings thereby expressed at different time periods, that is the understanding of the relationship between operations, present both similarities and differences.

[20] In fact, the *Mathematical Classic by Master Sun* contains texts for procedures executing multiplications, divisions and square root extractions. It describes explicitly the positions and the basic operations on them required to execute these arithmetical operations. Clearly, during the process of execution of a root extraction as described in the *Mathematical Classic by Master Sun*, the positions named “dividend,” “quotient,” “divisor,” undergo transformations that can be correlated with those the positions having the same names undergo in the process executing a division. The same conclusion holds true for *The Nine Chapters,* even though the actual correlation displayed changes. In other words, the practice is the same, even though what is expressed changes. In K. Chemla, "Cas d'adéquation entre noms et réalités mathématiques. Deux exemples tirés de textes chinois anciens", in *Le juste nom*, eds. Karine Chemla and François Martin, *Extrême-Orient, Extrême-Occident* (Saint-Denis, 1993), I interpreted an assertion by Li Chunfeng in his commentary on *The Nine Chapters* as referring precisely to both this commonality of practice and the difference of meaning expressed. Let me emphasize that it is the practice of naming positions (位 *wei*) in this context that is more specifically the point upon which Li Chunfeng’s commentary bears. The point will prove meaningful below. What is also important, in this context, is that in the *Mathematical Classic by Master Sun*, we further meet with the same phenomenon for the processes of a multiplication and a division, except that in these cases, the relationship displayed between the processes executing the operations on the computing instrument is that of an opposition, and not that of a correlation. However, in this case, the *Mathematical Classic by Master Sun* inserts one of these extremely rare second-order comments, which provides evidence of actors’ awareness of the practice. We read at the beginning of the text for division: “The method for any division is *exactly opposed* to that of multiplication” (Qian 1963, vol. 2: 282, my emphasis). This statement can precisely be read as referring to what is expressed by means of the practice of writing down processes of computations on the computing instrument. If we recapitulate what we have seen so far, *The Nine Chapters* and the *Mathematical Classic by Master Sun* have several features in common. On the one hand, they share the same algorithm for the operation of division, designated in the two cases by the term *chu*. On the other hand, they both bear witness to the same practices of expressing the relationship between operations, by means of texts of algorithms executing them as well as a material inscription of the processes on the computing instrument. Let me stress once more that, despite these continuities in the practices, the relationships expressed differ.

The Nine Chapters provides evidence showing how this practice was put into play for square and cube root extractions. The *Mathematical Classic by Master Sun* bears witness to how the same practice expressed the relationship between multiplication, division and root extraction. We have seen above that *The Nine Chapters* indirectly refers to an algorithm for division similar to the one described in the *Mathematical Classic by Master Sun*. On this basis, we can also observe that the texts of the algorithms in *The Nine Chapters* manifest the fact that the same type of material relationship was established on the computing instrument between the process executing a division and the one executing a square (resp. cube) extraction. This observation yields in fact a key resource to supplement the information given by the texts of *The Nine Chapters* and restore completely the process of computation on the surface, not only for division, but also for root extractions. To recapitulate with respect to the topic of this chapter, so far, the key practices thanks to which we could grasp knowledge with respect to these three operations indirectly indicated in *The Nine Chapters* included practices of naming, practices of text as well as practices of computing.

Again with the means of expression that the computing instrument offered, division appears, in *The Nine Chapters*, as the fundamental operation from which root extraction derived, through the use of auxiliary positions. In conclusion, the structure of the set of operations to which the terminology prescribing the operations refers is parallel to the structure that is expressed, on the one hand, by the relationship between the texts of algorithms given to execute the operations and, on the other hand, by the relationship between the processes executing the operations on the computing instrument. We see that the knowledge actors displayed by these means includes not only knowledge of the relationships between the operations, but also knowledge of the inner structuring of each of the processes of computation executing them. The insight gained into actors' knowledge thanks to a description of their practices is manifest. Further, this insight will allow us, in what follows, to suggest how this knowledge appears to have served as a basis for further developments.

Still in relation to the computing instrument, some of the features of the texts of the algorithms in *The Nine Chapters* give us clues regarding the number system on which the calculations were executed. Through some of the actions prescribed (such as jumping columns, moving values forward and backward), through the iterative structure of the texts, we know that the algorithms operated on a place-value decimal system, in a context in which numbers were written down with counting rods. Again in this case, we see how our sources indirectly shed light on elements of actors' knowledge —and practice— that are not a topic of explicit discursive treatment, at least in the remaining set of documents. In conclusion, it appears that not only the algorithm for division indirectly manifested is the same as the one described by the *Mathematical Classic by Master Sun*, but the number system tacitly assumed by *The Nine Chapters* is also the same as the one explicitly outlined by this other Classic, composed a few centuries later. In addition, the two classics share practices of naming, practices of writing down texts for algorithms as well as practices of inscribing processes of computation on the computing instrument. This should not surprise us: a number system, the algorithms based on it as well as practices related to them form a coherent system, and its cohesion has consequences on how the system is transmitted.

II. 3 *Quadratic equation as an operation in* **The Nine Chapters**

Problem 19 of chapter 9 in *The Nine Chapters* is solved by a procedure that is concluded by an operation.[21] The statement of the last operation of the procedure is characterized by the fact that its operands are designated by technical terms, that is, "dividend" and "joined divisor." We meet again with the correlated situation that operands are associated with technical terms and that we are within the framework of division. The latter conclusion is confirmed by the fact that the operation is then prescribed by the verb *chu* "divide," prefixed by a qualification. However, the identity of the operation is not immediately clear. Indeed, its prescription reads "divide this by extraction of the square root *kai fang chu zhi* 開方除之." We thus have the formulation usually referring to a square root extraction, as an operation deriving from division. However, surprisingly enough for us, the operation is *not* a mere square root extraction: instead of having a single operand, as should be the case, in the context of the procedure the operation designated by means of the same technical term is now applied to two operands, a "dividend," and also a "joined divisor." In what follows, we shall call them, for the sake of simplicity, *a* and *b,* respectively. The procedure solving problem 19 thus describes how to compute the value of both operands *a* and *b*, before concluding by the prescription "divide this by extraction of the square root *kai fang chu zhi* 開方除之." Apparently, the set of technical terms attached to the operation indicates that a new operation is put into play. What is it?

Before we interpret the nature of the operation and the reasons why the prescription takes this form, a remark imposes itself on us. To formulate this remark, let us begin with an outline of the conditions in which we operate to interpret the text. *The Nine Chapters* contains only two pieces of text on which we can rely to determine the nature of the operation. On the one hand, we have the text of an algorithm prescribing how to execute a square root extraction. On the other hand, we have problem 19 of chapter 9, and the related procedure bringing the operation into play. Nothing more in the Classic. Naturally, we also have the commentaries on *The Nine Chapters* and other writings, which make use of the same or of similar operations. Let us leave them aside for the moment. Where does the difficulty in the interpretation of the operation stem from? In fact, in my view, it derives from the fact that we have an expression prescribing the operation as if it were, and at the same time were not, a square root extraction. Suppose the terms of the operation had been called xxx and yyy, suppose the operation has been prescribed by the term zzz. We would have done what we do for addition or multiplication, without thinking about it. We know the operation to be applied to xxx and yyy that the solution of the problem requires at this point. Therefore, we would conclude that zzz must refer to this operation, no matter how the operation was executed. However, such was not the terminological choice to which *The Nine Chapters* bears witness. The authors opted for a completely different set of terms, which derives from their practice of technical terminology. Their terms not only *refer* to an operation, but here like above, they also *formulate* a state of understanding regarding the relationships between various operations. In our case, they assert an intimate connection between our operation and, respectively, division and square root extraction.

[21] Li Jimin 李繼閔, *Dongfang shuxue dianji*: 112-114. The text and an interpretation can be found in Chemla and Guo Shuchun, *Les neuf chapitres*: 671-672, 698-699, 734-735 and related footnotes.

If we were confusing these two parts played by the terminology in this context with one another, this could create a problem for the interpretation of the text. Interpretation would indeed become problematic, if, instead of reading the prescription "divide this by extraction of the square root *kai fang chu zhi* 開方除之" as *stating* a relationship between the operation meant and root extraction, we were expecting that the expression merely refers to an operation. We would then be led to wonder why an operation which is *not* a square root extraction can be prescribed as if it were, or whether the operation prescribed is actually a separate operation. In my view, problems of interpretation of that kind occur precisely when we do not pay attention to actors' practices in the context observed —here, their practice of terminology—and when, instead, we implicitly assume their practices as identical to ours. The problem vanishes if, by contrast, we base our treatment of the sources on a description of the practices, in the context of which they were produced. In addition, this description allows us, as we shall explain in greater detail, not to miss meanings expressed by the choice of terms. These two facets echo the two parts played by the specific use of technical terms.

In the case examined, the difficulty is compounded by two facts. As we shall see, the operation meant is not what we consider an ordinary arithmetical operation. Further, it is a mathematical entity that for us does not have the identity of an operation. Such are the obstacles to overcome, I claim, to get the interpretation of the text for which I argue.

Finally, still before we turn to identifying the operation meant, let me stress yet another point, important for my purpose in this chapter. The case examined here does not only show, once more, why the description of practices is essential to interpret documents, but it also illustrates how such a description puts us in a position to perceive the work actors carried out in that context. In other words, what is at stake here is not merely interpreting the operation meant or understanding, as was the case above, the relation actors established between that operation and other mathematical entities. What is further at stake is to bring to light an effort actors devoted to the study of operations and the knowledge thereby gained with respect to them. The reason why these three features should not be separated is simple. The understanding that actors' terminology reflects derives from their musings on operations. One of the outcomes of this work, that is, a conception of the structure of a set of operations, can be captured in *The Nine Chapters only* by analyzing the terminology. In fact, it is *because* there was a work on the set of operations and *because* the results of this work are reflected in the terminology that we, as present-day exegetes, have trouble in interpreting the operation. Now, we can look at the situation from another angle. If we are to account for actors' knowledge about the operations, we are not only interested in the operations they "knew" —the answer to the question of the interpretation—, but also in how they explore them and understood them.

As a result of the previous analysis, it appears that we need to dissociate the analysis of the operation concluding the solution of problem 19 of chapter 9 into two steps. We must first identify the operation referred to, as if it was named zzz. We must then interpret which results can be gleaned from the structure of the terminology.

Let us thus begin with the identification of the operation. If we examine the problem from a modern viewpoint, we understand that the operation corresponds to what in modern terms is a quadratic equation. That is to say, the execution of the operation is equivalent to solving the equation

$$a = x^2 + bx.$$

What for us is an equation thus presents itself in *The Nine Chapters* in the form of a numerical operation, like division or root extraction.[22] I shall call this entity an “operation-equation.” This fact may be surprising, but we must be prepared to acknowledge that for entities that we recognize as same, actors of the past have shaped concepts that differ from ours. Here, the interpretation may sound all the more surprising that the operation-equation is described as having only two terms (corresponding to the coefficient in x and the constant term), and not three (including the coefficient of x^2), as we would expect. Such an understanding of the operation-equation can be evidenced until at least the 11th century in China, when it is replaced by a new concept, for which the operation-equation has three terms. In addition to the two terms a and b mentioned above, the term which for us is the coefficient of x^2 is by then identified. This interpretation of the operation in *The Nine Chapters* is confirmed by Liu Hui’s commentary. In it, the commentator establishes the correctness of the procedure. In this case, he shows how a square of side x, the unknown, and a rectangle of area bx make together the area a. His commentary ends on that point. He has thus shown that the “operation” was proved to conclude correctly the procedure, since it amounted to stating the relationship mentioned. So much for the identity of the operation. What can we say about its execution?

Liu Hui adds nothing about how the operation-equation should be executed. Nor do the authors of *The Nine Chapters*. The prescription of a square root extraction appears to be sufficient in their eyes for the user of the procedure attached to problem 9.19 to know what to do. Observing some features of the practices to which the text of this algorithm adheres, and the execution of the extraction on the computing instrument, enables us to put forward a hypothesis about this facet of the operation-equation. Suppose the algorithm for extracting square roots, described in *The Nine Chapters*, be applied to the number A. It determines the root digit by digit, beginning with the digit corresponding to the highest order of magnitude of the base 10. Let us represent this part of the root by the modern expression $p.10^n$. The first steps of the algorithm in *The Nine Chapters* create on the computing instrument an array that we can represent in modern terms as follows:

Quotient	$p.10^n$
Dividend	A
Divisor	$p.10^{2n}$

[22] I have discussed the identity of various types of algebraic equations in the ancient world in Karine Chemla, "Nombres, opérations et équations en divers fonctionnements: Quelques méthodes de comparaison entre des procédures élaborées dans trois mondes différents", in *Nombres, astres, plantes et viscères: Sept essais sur l'histoire des sciences et des techniques en Asie orientale* eds. Isabelle Ang and Pierre-Etienne Will, Mémoires de l'Institut des Hautes Etudes Chinoises (Paris, 1994a). Moreover, I have devoted another article to the discussion of this concept of equation and its subsequent history in China: Chemla, "Changing mathematical cultures, conceptual history and the circulation of knowledge." I refer the reader to these two other publications for a more careful discussion of the interpretation of the text and, especially, its mathematical dimensions. I focus here on the knowledge of the structure of a set of operations to which *The Nine Chapters* testifies.

In the left column, I have indicated the technical terms borrowed from division, by means of which the text of the algorithm of root extraction refers to the entities. We recognize, in the lower row, the technical term introduced by the text describing the execution of the algorithm. This term does not refer to any operand of the root extraction. On such configurations, the operation of "division *chu*" multiplies the digit of the "quotient," p, by the "divisor," and subtracts the product from the "dividend." The operation yields in our case the following array:

Quotient	$p.10^n$
Dividend	$A — p^2.10^{2n}$
Divisor	$p.10^{2n}$

In order to prepare the determination of the next digit, the algorithm prescribes transformations to be applied to the "divisor," yielding

Quotient	$p.10^n$
Dividend	$A — p^2.10^{2n}$
Fixed Divisor	$2p.10^{2n-1}$

The important point here is that, if we forget the first phase of the algorithm of root extraction —the one described above—, the operation whose execution starts at this point solves the following quadratic equation[23]

$$x^2 + 2p.10^{2n-1}. x = A — p^2.10^{2n}$$

which is written on the computing instrument as follows:

Quotient	
Dividend	$A — p^2.10^{2n}$
Fixed Divisor	$2p.10^{2n-1}$

We see that there remain on the surface precisely the two terms that are the operands of the operation-equation, which concludes the procedure of problem 19 in chapter 9. We also observe that the technical terms designating the operands of this operation-equation in the procedure are correlated to those designating the values in the rows represented above.[24] Let us draw some conclusions about this situation.

First, it appears that a subprocess of the execution of a root extraction was detached from this context and given the identity of an operation. This conclusion about the origin of this type of "quadratic equation" explains why the operation-equation was perceived as having only two operands. It also elucidates why the prescription could be made by using the expression "divide this by extraction of the square root." It lastly explains why *The Nine Chapters* provides no new algorithm prescribing how the operation-equation should be executed. Seen from the viewpoint of the execution of the operation, the operation "quadratic equation" depends on the square root extraction in

[23] At the time in China, such an equation is conceived as having a single root. I do not enter further into these details, useless for my purpose.
[24] To avoid adding useless details here, I do not comment on the difference of terminology between "fixed divisor," "divisor" and "joined divisor."

that it is executed by a subprocedure. This type of link between operations differs from the link evidenced above between root extractions, or between the latter operations and division. “Surgical operations” of that kind can be shown to fit the practice with the computing instrument at the time. Again, the description of the practice helps us inquire into features of the knowledge on which the sources remain silent.[25]

Second, we have emphasized above how an entity that was *not* an operand of root extraction, the “divisor,” was nevertheless identified as a technical component of the process of execution and even given a name. We now see further that this fact is correlated with how a new operation was created by derivation from root extraction.

Let us now consider the knowledge that is reflected in the choice of terminology and more generally recapitulate the facts we have surveyed. We have seen that in *The Nine Chapters* four operations are mentioned, all of which are related to each other at several distinct levels. The foundation of the set is constituted by the operation of division. Its name *chu* occurs in expressions prescribing all the other operations. Its process of execution is the basis by reference to which the executions of root extractions are written down and carried out on the surface. Further, the execution of root extraction provides in turn source material for the derivation of a new operation, i.e., the operation-equation, or “quadratic equation.” The kind of relationship established between the operations is not the same, depending on the case. However, they form a group that presents, in *The Nine Chapters*, quite a precise and complex structure. In fact, the terminology prescribing these operations has a structure that is transparent on the relationships that the texts of algorithms or their execution shape and state in their way.

II. 4 *The work done with operations as reflected in* The Nine Chapters

So far, we have relied on clues about operations that could be found in *The Nine Chapters* and related texts like the *Mathematical Classic by Master Sun*. These clues included names used to refer to operations or operands, and texts for algorithms executing the operations. What we have shown is that these clues bear witness to a work that has been done on the operations and the relationships between them. They testify to a knowledge about operations that can be shown to be possessed by the authors of the procedures recorded in *The Nine Chapters*. One of the outcomes of this work, which is one of the components of the knowledge identified, consists of a conception of the structure of the set of four operations.

Bringing such a work and its outcomes to light is essential for history of mathematics. Indeed, it suggests that operations were not only tools in the ancient world, but also objects of study. Our analysis allows us to perceive the results of theoretical inquiry into operations.

Since the pieces of knowledge acquired were not the topic of discursive developments in our sources, it was only through relying on observation of various aspects of mathematical practice (practices of naming, of writing down texts for algorithms and shaping dynamic inscriptions on the computing instrument) that we could derive our results from the clues found in the sources. Our understanding of the mathematical practice in relation to which *The Nine Chapters* was composed in ancient

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[25] Chemla, "Changing mathematical cultures, conceptual history and the circulation of knowledge", gives an argument about how concretely the operation was detached. In this other article, I emphasize *how* in the subsequent centuries, knowledge about algebraic equations in China appears to have been sought for within the same conceptual framework.

China was essential to help us perceive knowledge in the background of our sources. It is in this sense that the description of mathematical practice can provide tools to carry out conceptual history in a new way, as I suggested at the beginning of this chapter.

In fact, the body of knowledge uncovered shows strong correlations with what we see in later documents from China. Clearly it represents a stage in the understanding of operations that is reflected in *The Nine Chapters* and that served as a basis for further later developments. As we have already alluded to, algorithms for the execution of root extraction remained a topic of work for centuries to come. This work was carried out by means similar to those examined above and it regularly reshaped the relationships between division *chu* and root extractions that *The Nine Chapters* formulated. Moreover, knowledge about algebraic equations was developed in subsequent centuries in a similar conceptual framework. Further, the division *chu* appears to have played a central role in theoretical work done on operations in subsequent centuries. Together with the opposed operation of multiplication *cheng* 乘, it proved central in practices of proving as well as in the inquiry to which several texts bear witness and that aimed at uncovering the most fundamental operations.[26] These remarks retrospectively support our interpretation of the knowledge possessed by those who produced the text of *The Nine Chapters*, even though they did not formulate it discursively. Moreover, they cast further light on the essential part played in this history by the division *chu*, which, in what follows, will prove a key fact. The same remarks show from another angle how conceptual history is enriched, when one deals not only with explicit knowledge recorded in our mathematical sources, but also with the clues they contain.

However, the results stated so far also leave some questions unanswered. The first question that must be addressed, if we want to ascertain that the clues in which we read meanings are not the result of chance, is the following: Of which kind of history is the complex of operations described, that is, the body of knowledge outlined above, an outcome? How can we capture, from a historical viewpoint, the mathematical work whose results we perceive on the basis of our reading of *The Nine Chapters*? Can we recover which agenda on operations were pursued? One may also want to understand how the ways of working on operations took shape, that is, inquire into the history of practices with operations. Given our focus in this chapter, we shall leave the latter issues aside to concentrate on the former questions raised.

In fact, in the last decades, new mathematical manuscripts from early imperial China, i.e., produced in the 3rd and 2nd centuries BCE, were discovered. The full text of two of them was recently published. These publications allow us to start addressing the question of the historical process, through which the body of knowledge regarding operations as evidenced in *The Nine Chapters* took shaped. The sources are more or less of the same nature as *The Nine Chapters*. Consequently, it will again be an examination of clues found in the sources, and the practices they betray, that will enable us to draw some conclusions.[27] Quite surprisingly and unexpectedly, what these new documents

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[26] Guo Shuchun 郭書春, *Gudai shijie shuxue taidou Liu Hui 古代世界數學泰斗劉徽 (Liu Hui, a leading figure of ancient world mathematics)*, 1 ed. (Jinan, 1992): 301-320, Karine Chemla, "Mathematics, Nature and Cosmological Inquiry in Traditional China", in *Concepts of Nature in Traditional China: Comparative Approaches*, eds. Guenther Dux and Hans-Ulrich Vogel (Leiden, 2010).

[27] In what follows, for the sake of simplicity and in order to concentrate on the topic of the chapter, I state results I have obtained, without presenting the full arguments on which they are based. I shall provide them in other publications.

show is a mathematical landscape quite different from what was uncovered above. This is what I shall now show.

III. Capturing theoretical work carried out on division in ancient China

The previous section argued that *The Nine Chapters* bears witness to a specific knowledge with respect to a set of operations. This set includes division, square and cube root extraction, as well as an operation-equation (i.e., a form of quadratic equation.) The purpose of this section is now to show that the indirect evidence the earliest extant texts provide about these operations before the time of *The Nine Chapters* indicates a different state of knowledge with respect to the same operations. The conclusion that we shall then derive is that we can get some idea —very rough indeed, but still some idea— about the time period when this body of knowledge took shape. More importantly, we can suggest hypotheses about the process through which the complex of operations described took shape.

Let us begin by presenting the documentary evidence we have. The first manuscript that was found, the *Book of mathematical procedures* (筭數書 *Suanshushu*), was excavated in 1984 from a tomb at Zhangjiashan 張家山 (Jiangling county, Hubei province).[28] Peng Hao gives *ca* 186 BCE as a *terminus ante quem* for its composition. Since 2007, two new mathematical manuscripts have been brought to light. The earliest of the two books found dates from the Qin time period (third century BCE), and actors entitled it *Shu* 數 (*Mathematics*). Its entire text was recently made public.[29] This manuscript, the product of illegal excavation, was bought in December 2007 on the Hong Kong antiquities market. By contrast, the second mathematical book that was found, entitled 算術 *Suanshu* (*Mathematical procedures*), was excavated at Shuihudi (Yunmeng county, Hubei province) in the context of regular excavations. The archeologists working on this source material argue

[28] A first critical edition with annotations was published in PENG Hao 彭浩, *Zhangjiashan hanjian «Suanshushu» zhushi 張家山漢簡《算數書》注釋 (Commentary on the Book of mathematical procedures, a writing on bamboo strips dating from the Han and discovered at Zhangjiashan)* (Beijing, 2001). It was first translated into Japanese in Jochi Shigeru 城地茂, "Sansusho nihongo yaku 算數書日本語譯" (Japanese translation of the Suan shu shu)", *Wasan kenkyuso kiyo 和算研究所紀要* 4 (2001). We have mentioned above the two translations into English that have appeared: Cullen, *The Suan shu shu 筭數書 'Writings on reckoning'* and Dauben, "算數書. Suan Shu Shu". A critical edition and new translation into Japanese and Chinese have also appeared: 張家山漢簡『算數書』研究会編 Chôka zan kankan Sansûsho kenkyûkai. Research group on the Han bamboo strips from Zhangjiashan *Book of Mathematical Procedures*, *漢簡『算數書』Kankan Sansûsho. The Han bamboo strips from Zhangjiashan Book of Mathematical Procedures* (京都 Kyoto, 2006).

[29] See the Ph.D thesis by Xiao Can 肖燦, "Research on the Qin strips "Mathematics" kept at the Academy Yuelu 嶽麓書院藏秦簡《數》研究" (History, Hunan University 湖南大學, 2011) as well as Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 主編, eds., *嶽麓書院藏秦簡（貳）Qin Bamboo slips kept at the Academy Yuelu (2)* (2011).

that the book was copied at the beginning of the Han dynasty in China, before 157 BCE. We still wait for their publication of the text.[30]

The evidence that the manuscripts produced by archeology provide differs in nature from what we know through texts like *The Nine Chapters*, which were handed down through the written tradition. On the one hand, this fact relates to the different modes of transmission and the correlated kinds of changes that the books could undergo in these different channels of transmission. On the other hand, the mathematical books from ancient China handed down through the written tradition were for the most part passed on with ancient commentaries attached to them. When these commentaries have survived, they help us deal with the evidence provided by the book.

In what follows, it will also be useful to take into account the testimony of another book that was handed down and that was apparently composed before *The Nine Chapters*, that is, *The Gnomon of the Zhou* (*Zhou bi* 周髀).[31] Different opinions have been put forward regarding the date of the completion of the book. Some scholars date *The Gnomon of the Zhou* from the 1st century BCE (Qian Baocong considers it dates from *ca* 100 BCE), whereas others consider it was completed in the early 1st century CE (C. Cullen). This document gives us information regarding mathematical knowledge and practices used in the context of astronomical activity. In 656, together with *The Nine Chapters* and other writings, *The Gnomon of the Zhou* was selected to enter an anthology of Classics, *Ten mathematical Classics*. In relation to its status as classic, commentaries on it had been composed in the preceding centuries and, in 656, some of them were selected to be handed down with *The Gnomon of the Zhou.* We shall mention below the commentary on this classic that Zhao Shuang composed in the third century.[32]

What do these other writings show, by comparison with *The Nine Chapters*?

The first prominent fact to which all documents older than *The Nine Chapters*, including *The Gnomon of the Zhou*, testify is the difficulty that the execution of divisions presented at the time.

On the basis of my interpretation of the texts, an important part of the *Book of mathematical procedures* as well as *Mathematics* is devoted to addressing problems caused by divisions.[33] The difficulties included, first and foremost, dividing quantities expressed with respect to different systems of measuring units (capacity, volume, areas,

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[30] Compare Karine Chemla and Biao Ma, "Interpreting a newly discovered mathematical document written at the beginning of Han dynasty in China (before 157 B.C.E.) and excavated from tomb M77 at Shuihudi 睡虎地", *Sciamvs* 12 (2011).

[31] Qian Baocong 錢寶琮, *Suanjing shishu (Qian Baocong jiaodian)* contains a critical edition of the book. It was the basis for the translation into English published in Christopher Cullen, *Astronomy and mathematics in ancient China: the Zhou bi suan jing*, ed. Christopher Cullen, *Needham Research Institute studies ; 1* (Cambridge [England]; New York, 1996). In what follows, I shall also rely on the same critical edition.

[32] The anthology provided the editions of books that were used as textbooks to teach mathematics, in the context of the College of Mathematics (*Suan xue* 算學), and prepare candidates for the newly established state examinations on this topic. Compare Man-Keung Siu and Alexei Volkov, "Official Curriculum in Traditional Chinese Mathematics: How did Candidates Pass the Examinations?", *Historia Scientiarum* 9 (1999).

[33] Another reflection of the same fact could be the importance granted to the expression and treatment of proportions, which, according to PENG Hao 彭浩, *Zhangjiashan hanjian «Suanshushu» zhushi*: 17-19, represents one half of the *Book of mathematical procedures*.

weight and so on) and including fractions of measuring units.[34] The problems also included apparently the process of dividing itself, when actors had to divide quantities expressed with respect to systems of measuring units and yield a result of the same type.

In fact, the only mathematical topics on which *The Gnomon of the Zhou* gives lengthy developments are precisely these two aspects of division.[35] Accordingly, the difficulties attached to dividing are reflected in the wealth of details given about the overall execution of a division in *The Gnomon of the Zhou*. The procedures carrying out the division of quantities expressed with respect to various measuring units, first, show how to transform dividend and divisor into abstract integers, so that the quotient yielded corresponds to a determined measuring unit. They then prescribe how to decompose the execution of the division into a sequence of elementary divisions, each of which yielding the component of the quotient corresponding to a given order of magnitude. To do so, the successive remainders are transformed, before being, in turn, divided by the same divisor, so as to obtain the related part of the quotient. In the end, bringing these results together yields the global result of the division in the form of a quantity expressed with respect to a series of measuring units and orders of magnitude. In this context, *The Gnomon of the Zhou* provides key information, on the basis of which we shall be able to derive the clues we need and conclude. Let us thus consider in greater detail a piece of evidence that the book contains.[36]

A length, expressed as an integral number of the measuring unit *li*, is to be divided by a number of days, which consists of an integer increased by a fraction. In a first step, both values are correlatively transformed into abstract integers yielding the same result (952000 and 1461), before the first division be prescribed. This first prescription is carried out as follows: "(…) makes the dividend (*shi*), (…) makes the divisor. Eliminating this (*chu zhi* 除之), (each time it is) like the divisor (*ru fa* 如法*)*, it yields one *li* (*de yi li* 得一里). " I interpret here the same character *chu*, which above was interpreted as meaning "to divide," as referring now to a subtraction. Accordingly, I translate *chu* in this context as "to eliminate." In other words, I claim that between the two contexts, this key term has changed meaning. I shall come back to this suggestion and the overall expression prescribing division below. The prescription makes clear the measuring unit to be associated with each unit obtained for the quotient, each time a quantity equal to the divisor is subtracted. The formulation evokes, as we have already seen in the previous section, one of the expressions by which *The Nine Chapters* refers to a division. Here too, we shall come back to this point.

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[34] Some of the procedures shaped to address this issue are of the type discussed in Karine Chemla, "Documenting a process of abstraction in the mathematics of ancient China", in *Studies in Chinese Language and Culture - Festschrift in Honor of Christoph Harbsmeier on the Occasion of his 60th Birthday*, eds. Christoph Anderl and Halvor Eifring (Oslo, 2006). In historians' publications on the manuscripts, since these documents were released, the importance of these procedures has been so far underestimated. I shall come back to them more systematically elsewhere. The difficulty posed by division is also reflected by the number of procedures given to divide between fractions or integers increased by fractions.

[35] See, for instance, Qian Baocong 錢寶琮, *Suanjing shishu (Qian Baocong jiaodian)*: 52, 61.

[36] I rely on Ibid. , vol. 1 : 52. The same comments hold true for the passage mentioned above, on p. 61.

What is most important for us now is the following part of the execution of the division. At this point, one has obtained 651 *li* and there remains 889 in the dividend, which is smaller than the divisor 1461. The execution goes on, in order to obtain the remaining part of the quotient, expressed with respect to the measuring unit for length that is smaller than the *li*, that is, the *bu*. A *li* is 300 *bu*. If the practice of computation was the same as the one attested to later, the remainder 889 would be multiplied by 300, and the product would then be divided by the divisor 1461 to yield the additional number of *bu* in the quotient, 182 *bu* and 798/1461 *bu*. This is not merely my own perception. This also is what the 3rd century commentator Zhao Shuang expects, when he repeatedly begins his commentary on the related segment of the computation as follows: "One should multiply by 300…." However, in *The Gnomon of the Zhou* the subsequent part of the operation is not executed in this way, which gives important clues.

In contrast to Zhao Shuang's expectation, the remainder is first multiplied by 3, computing what Zhao Shuang interprets as "the *dividend of the hundreds* (*bai shi* 百實)" (my emphasis). *The Gnomon of the Zhou* then prescribes a second division of this dividend by the same divisor. The prescription is formulated as follows: "(each time it is) like the divisor (*ru fa* 如法), it *yields a hundred bu* (得百步 *de bai bu*)" (my emphasis). This division yields the value of the part of the quantity sought-for that corresponds to the hundreds of *bu*. The remainder of that division is then multiplied by 10, determining what Zhao Shuang in turn calls "the dividend of the tens." After this transformation of the dividend, *The Gnomon of the Zhou* prescribes a third division by means of the following expression: "(each time it is) like the divisor (*ru fa* 如法*), it yields ten bu* (*de shi bu* 得十步)" (my emphasis). Lastly, the same procedure is repeated to determine the units in the part of the quotient which is expressed in *bu* and it is concluded by the following statement : "(each time it is) like the divisor (*ru fa* 如法*)*, it yields one *bu* (*de shi bu* 得一步). The (part of the dividend) which does not fill up the divisor, it is named by the divisor."[37]

What clues can we derive from this way of practicing the division?

First, the procedure reflects features of the number system on which it is based and about which we have no direct explicit information. It also casts light on features of the computing device used. As was alluded to above, *The Gnomon of Zhou*, like the newly discovered manuscripts, clearly refers to computations carried out on a surface, on which numbers are represented with counting rods. Some of the quantities mentioned in the text of the procedure sketched above appear to have their values modified throughout the calculation. This feature confirms that they were written in a way allowing these changes. Rods do fit this expectation. However, this first conclusion does not suffice to allow us to determine the kind of number system on which the procedures operated at the time. We should be careful before assuming anything further about the practice of calculation.

In fact, the procedure from *The Gnomon of Zhou* examined fully justifies this caution. [38] Even though it is clear that the procedure makes use of a decimal conception of the quantity, in the production of the amount of *bu* in the result sought-for, the way in

[37] The last sentence is the usual prescription when the remainder of the dividend is taken as numerator of a fraction, whose corresponding denominator is the divisor.
[38] Note that a procedure containing steps similar to those on which we focus in *The Gnomon of the Zhou*, in the context of a decimal system of measuring units, is evidenced in the *Book of mathematical procedures*, slip 42, PENG Hao 彭浩, *Zhangjiashan hanjian «Suanshushu» zhushi*: 56.

which the calculation is conducted makes it unlikely that it relies on a place-value notation, like we have seen was the case for *The Nine Chapters*. Is this conclusion correct? This question incites us to mine our corpus of manuscripts and look for clues it could disclose about the number system. The first important conclusion is that, as far as I can tell, we have in these early writings no positive trace that a place-value system was used, like we have in *The Nine Chapters* and later texts. No procedure, for instance, seems to bring into play the elementary operations of moving quantities forward or backward, in order to multiply or divide them by a power of 10.[39] By contrast, as was mentioned in the previous section, these operations occur frequently in later texts, exploiting the fact that they operate on the basis of such a number system. Observing the practice of algorithms, and the operations they put into play, thus gives us insight into the number system used.

If this is correct, it suggests that a major change occurred between, on the one hand, the time of the production of the manuscripts and the composition of the procedures examined in *The Gnomon of the Zhou*, and on the other hand, the time when *The Nine Chapters* was compiled. The conclusion is supported by another indirect piece of evidence about the practice. The manuscripts contain tables of multiplication between powers of 10, which would seem useless in case the number system had the property of being place-valued.[40] In fact, such tables disappear from mathematical writings such as *The Nine Chapters*. As a preliminary conclusion, our first set of clues thus suggests that the number system in the context of which such divisions were carried out was decimal, but not place-valued. Accordingly, rods may have been used on the surface in a way different from what we have outlined in the previous section.

The second set of clues we can derive from the procedure of *The Gnomon of the Zhou* that we have described above in some detail comes from the terms used to prescribe divisions. To begin with, let us notice that the context of this procedure makes clear the origin and meaning of the expressions that we find in all our writings, that is, "when the dividend is like the divisor, then one" or "then it yields one" or else "then one (name of a measuring unit)" (*shi ru fa er yi* 實如法而一， *shi ru fa de yi* 實如法得一， *shi ru fa er yi* (name of a measuring unit) 實如法而一(name of a measuring unit)). The clarification comes from the fact that now these expressions are found in a context in which we have other expressions such as "(each time it is) like the divisor (*ru fa* 如法), it *yields a hundred bu* (得百步 *de bai bu*)" (my emphasis) or "(each time it is) like the divisor (*ru fa* 如法), *it yields ten bu* (*de shi bu* 得十步)" (my emphasis). All these expressions seem to reveal a focus on the meanings of, that is, the orders of magnitude of, and units to be attached to, the successive parts of the result, whether these be hundreds of *bu*, tens of *bu* or units *bu*. This emphasis on the determination of the meanings of the units produced by the operation on the dividend and the divisor maybe indicates the difficulty caused by the fact that the operands were modified in several ways during the execution of the division. Such expressions possibly also cast light on the actual procedure of division used, that is, one based on successive subtractions. In

[39] For instance, the procedure of *The Gnomon of the Zhou* prescribes twice to multiply by 10, to yield first what Zhao Shuang interprets as "the dividend of the tens" and then "the dividend," which corresponds to the units. In neither case, is the multiplication prescribed as a displacement of the rods. Exactly the same feature characterizes the procedure in the *Book of mathematical procedures* mentioned in the preceding footnote.
[40] Compare Chemla and Ma, "Interpreting a newly discovered mathematical document".

fact, the expressions used in the manuscripts and in *The Gnomon of the Zhou* are all similar in inspiration.[41]

In relation to these remarks, it is striking that the verb *chu* does occur in these texts, but in these early writings, it never refers to division.[42] There *chu* only means "subtraction." Incidentally, it is the observation that has led me to interpret the first prescription of division in *The Gnomon of the Zhou* as I did, namely, "Eliminating this (*chu zhi* 除之), (each time it is) like the divisor (*ru fa* 如法*)*, it yields one *li* (*de yi li* 得一里). " In fact, this use of *chu* as "subtraction" in the statement of division does occur in the manuscripts, but never in *The Nine Chapters*. This remark shows the strong relation that exists between *The Gnomon of the Zhou*, that is, the earliest mathematical text handed down, and the earliest known mathematical documents, which were produced by archeology. The conclusion echoes, and gives support to, the assumption that we derived above and according to which these early writings share a similar number system, different from the one to which *The Nine Chapters* refers.

We have reached a crucial point: as far as the extant evidence allows us to draw conclusions, we see that the verb *chu*, which refers to division in *The Nine Chapters*, and constitutes the pivot of the structure of the set of operations that we observed in the previous section, seems to have acquired this meaning of "division" between the last century BCE and the first century CE.[43] More precisely, its earlier meaning of

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[41] Xiao Can 肖燦, "Research on the Qin strips "Mathematics" ": 121-124, gives an overview on the expressions referring to division in the manuscripts. Guo Shuchun郭書春, "Shilun *Suanshushu* de lilun gongxian yu bianzuan 試論算數書的理論貢獻與編纂(On the theoretical achievements and the compilation of the *Book of mathematical procedures*)", *Faguo hanxue 法國漢學 (French Sinology)* 6 (2002): 525-527 surveys the expression of division in the *Book of mathematical procedures*. The statement on the expression of division in *The Gnomon of the Zhou* and the other claims made on the book in what follows are my own conclusions. They require a philological argument that I shall develop elsewhere, at the same time when I provide a full analysis of the expressions for division in early Chinese mathematical texts.

[42] A passage from *The Gnomon of the Zhou* seems to contradict this assertion (Qian 1963, vol. 1 : 77-79). It is thus interesting to read Zhao Shuang's view on this passage "非周髀本文。蓋人問師之辭。其欲知度之所分，法術之所生。 This is not from the original text of *The Gnomon of the Zhou*. Probably these are the words of someone asking a teacher. He wants to know the division of the *du* and that from which procedures originate." We shall come back to the use of the term *chu* in the various early mathematical texts in another publication.

[43] The meaning of "subtraction" *chu* occurs in *The Nine Chapters,* even though rarely. For instance, one encounters it in the procedures attached to problems 3.17 and 6.16. In the former case, the use of the term "there remains" immediately afterwards makes the meaning of *chu* clear. However, in the latter case, the gloss attributed to the third century commentator Liu Hui makes the meaning of *chu* as "subtraction" explicit, as if he perceived it as an archaism that could cause problems to the readers. By contrast, the commentator does not comment on any occurrence of *chu* as division. This fact indicates that in these other occurrences, the commentator interprets *chu* as referring to division. Moreover, it shows that for the commentator division is not perceived as a subtraction that would be repeated. The different uses of the same term are one among many hints that *The Nine Chapters* was produced by compilation, a topic on which again I shall come back in a future publication. In addition, my claim is that, among the mathematical

"subtraction," evidenced in the most ancient texts, gave way to that of "division." In fact, not only did the *term chu* change meaning, but the practice of computation yields clues showing that the *process* of division also underwent a key change. Without entering into detail, let me indicate the main clue. The algorithm of "division *chu*" between integers to which *The Nine Chapters* bears witness, and which, let me emphasize, we can restore on the basis of the observation of mathematical practices, proceeds through a progressive diminution of the dividend and a decimal shift of the divisor. The latter feature exploits the place-value feature of the number system. By contrast, the procedure for division to which *The Gnomon of the Zhou* testifies and that we have described in some detail above, does not refer to any change of the divisor, a point which the commentator Zhao Shuang regularly emphasizes in his commentary on the lengthy procedures describing divisions. These procedures only modify the dividend, alternately diminishing it and multiplying it. This execution of the division may have created the need to make clear at each step, as the terminology prescribing the successive divisions does, the nature of the part of the result obtained.

In conclusion, we see that our two sets of documents —the manuscripts and *The Gnomon of the Zhou*, on the one hand, *The Nine Chapters* and later texts on the other— reveal a change, which apparently —as far as we can tell— occurred between the two corresponding time periods. This change appears to have concerned several correlated features: the number system, the way of prescribing division by means of a verb or not, and the way of executing a division on integers. What is essential for us in relation to the topic of this chapter is that all these facts can be captured by means of indirect reflection of what the practices of computation evidenced in the books show.

A last set of clues will allow us to conclude the chapter. It deals with square root extraction. The manuscripts contain procedures to find out the side of a square. However, first, the terms referring to this operation vary.[44] Second, none of these terms refers to the division *chu* or to an algorithm for division. Lastly, the procedures are different among themselves and different from division. In particular, in contrast to the procedure included in *The Nine Chapters*, they do not give any clue that they would rely on a place-value number system.

A fourth correlated feature can thus be added to the set of changes, listed in the previous paragraph and evidenced in *The Nine Chapters*. The manuscripts seem to indicate that algorithms for square root extraction were at the time not standardized. Further, they do not testify to any inquiry into the relationship of these procedures and

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documents extant at this day, the meaning of *chu* as "division" only occurs from *The Nine Chapters* on. Finally, the fact that commentators such as Liu Hui seem to perceive *chu* "subtraction" as an archaism is an important clue in favor of the idea that *The Nine Chapters* bears witness to a change in the understanding of the set of operations, when compared to the other earlier sources. More generally, mathematical sources from ancient China do not bear witness to independent traditions. This assertion does not mean, however, that the change evidenced in *The Nine Chapters* discussed in this chapter erase earlier bodies of knowledge.

[44] See, for instance, the procedure presented in the slips 185-186, entitled "Squaring a field (or finding the side of the square for a field) *fangtian* 方田," of the *Book of mathematical procedures* (PENG Hao 彭浩, *Zhangjiashan hanjian «Suanshushu» zhushi*: 124-125). Peng Hao notes (fn 1) that the algorithm differs from that of *The Nine Chapters.* Cullen, *The Suan shu shu 筭數書 'Writings on reckoning'*: 88, suggests that the latter procedure was perhaps not yet found at the time.

division that can be related to what later texts evidence. By contrast, in the state of knowledge to which *The Nine Chapters* bears witness, the name of the operation, the way in which square root extraction was executed and the way in which the text of the algorithm was written down all indicate a reshaping that brought square root extraction in close relation with the algorithm for the division *chu* and the place-value number system.

IV. Conclusion: Mathematical practices and conceptual history

Which conclusions can be drawn from the previous analyses? I shall answer this question from two perspectives.

To begin with, let us recapitulate the conceptual changes that we have brought to light with respect to operations in early imperial China. In section II, I have argued how an observation of various practices allowed us to perceive knowledge about operations in *The Nine Chapters*. It was less through interpreting sustained theoretical discourse, than it was through analyzing clues drawn from our sources, and interpreting them in the light of a description of related practices, that we could perceive a body of knowledge regarding operations. In particular, we have seen how at the time of *The Nine Chapters* the division *chu* appears to have been a key element in, and to have played a central part for, the set of known arithmetical operations. It was by reference to this operation that the texts for algorithms for square and cube root extraction were written down. The execution of these two operations on the computing instrument as well as the way of prescribing them also referred to division *chu*. In addition, a new operation, i.e., the quadratic equation, was introduced, on the basis of that kind of execution for square root extraction. According to the evidence derived from *The Nine Chapters,* knowledge about these four operations included knowledge about the structure of the set they formed.

These observations raised the question, which we addressed in section III, of understanding which historical processes led to the shaping of this knowledge. By relying on sources produced centuries earlier, we could suggest that the process was not that of a progressive derivation, on the basis of the division *chu*, of new operations, thereby naturally linked to it. On the contrary, we have shown that the operations of division and square root extraction were executed by other means in the time period before that of the composition of *The Nine Chapters*. They both underwent a key transformation.

The transformation appears to have been correlated to a change in the number system on which the algorithms were executed. Perhaps the place-value decimal number system, written down in China with counting rods, was introduced in this context. Moreover, the transformation not only reshaped the algorithms executing the operations but also established relationships between them, thereby giving rise to the structure that we brought to light. In this process, the operations were made to converge towards one another. Accordingly, these conclusions strongly support the claim that operations have been a topic of inquiry in ancient China, a fact that has remained unnoticed so far. Practitioners in ancient China thus did not limit themselves to the fact of creating means to execute operations. They also devoted some attention to the operations as such and shaped practices to work on them, thereby yielding new knowledge about their set. As a result, *The Nine Chapters* bears witness to a radical shift in the understanding of, and practice with, the four operations examined.[45] The shift is in particular clearly reflected in the new terminology that was introduced and that we have analyzed above.

I believe it is now clear how the observation of mathematical practices can provide new means to carry out conceptual history. In our case, it simply allows us to consider a history of the theoretical dimensions of the inquiry into arithmetical operations and of the means of working with these operations, which would otherwise, I claim, remain out of reach.

This remark leads me to my second set of conclusions, with respect, now, to the part that the description of practices can play in the history of science. Historiography of mathematics has mainly focused in the past on results stated and theories expounded by the actors. If we need to extend our interests and include the description of practices, it is not for the mere sake of describing them, but as an indispensible tool for a richer interpretation of our sources and a deeper inquiry into conceptual history. What I have endeavored to exemplify in this chapter is *how* the observation of practices —in the broad sense illustrated above—[46] can help us achieve such aims.

It is not by chance, I have suggested, that history of mathematics in the ancient world sheds light on the way in which our knowledge of practices can be a tool for a more fruitful mining of our sources. These sources are scarce and hence require elaborate treatments. Restoring practices in relation to which sources were produced allows us to dig out and interpret clues that indirectly give us insight into the results known to actors (e.g., how to divide) and more generally into actors' knowledge (e.g., a way of structuring a set of operations), when these bodies of knowledge only leave traces in the documents. Our knowledge of practices also allows us to detect clues of that kind.

What does the exercise now tell us about the opposition between practices and results, by means of which Léna Soler captures one of the meanings of the word "practice" in present-day science studies?[47] One of the outcomes of the argument developed in the chapter is to highlight in which ways the opposition between results or knowledge, on the one hand, and practices in this sense, on the other hand, may have to be rethought.[48]

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[45] One can also show that the reflection about division yielded, more or less during the same time span, other theoretical insights. Compare Chemla, "Documenting a process of abstraction". The use of terms like "divide in return *baochu* 報除" or the theoretical part devoted to the division *chu* more generally, for instance in the context of the algorithm "Measures in square" (*fangcheng* 方程), testify to yet other developments.

[46] Practices that I found useful to consider in this chapter include practices of writing texts for algorithms, including practices of naming operands and prescribing operations; practices of computing and of a computing instrument, including the structure of the number system it embodies. All these elementary practices shape a given mathematical practice. These pieces of information make clear in which sense I speak of mathematical practice in this chapter.

[47] I refer to the meaning 1 identified in Léna Soler, Etudier les pratiques scientifiques : étudier quoi ? A la place de quoi ? [website] (2012 [cited April 19 2012]); available from http://www.sphere.univ-paris-diderot.fr/IMG/pdf/LSoler_Practices_8Feb12.pdf.

[48] The aim of the conference "From practice to results in logic and mathematics," organized by the research group PratiScienS led by Léna Soler in Nancy, June, 21— 23, 2010 was to contribute to this program. The same issue is addressed in chapter IX of

To begin with, the case study analyzed in this chapter sheds light on the fact that, even in a field like mathematics, practices have their history. We could indeed perceive that in the main time periods examined above, the way of working with operations and executing them presented differences. For instance, in *The Nine Chapters*, clearly the layout of computations and elementary operations such as moving rods representing numbers forward and backward were important features of the practice of computing. In *The Gnomon of the Zhou* and the *Book of mathematical procedures*, the way in which divisions were conducted does not reveal any importance granted to these features. This remark about the historical dimension of practices has a direct bearing on our issue: the kinds of clues discussed in relation to our two sets of sources were of a completely different nature. In my view, as a matter of fact, practices are shaped in close relation to the questions addressed and they are shaped in the process of knowledge making. As a result both aspects of scientific activity are intertwined.

One echo of that intimate connection can be captured in the fact that, as we have seen, practices made use of knowledge about the topics explored. For instance, practices of computation rely on knowledge about the number system on which they operate. This is precisely one of the reasons why such practices can reflect these pieces of knowledge and evidence them.

In addition, the fact that practices and bodies of knowledge are shaped conjointly accounts for the fact that results and concepts can present correlation with practices. This idea is illustrated in this chapter by the example of quadratic equation. We have seen how the concept of quadratic equation to which *The Nine Chapters* testifies can be correlated to the practice of computation on the surface on which numbers were represented with rods. This concept differs from other concepts of quadratic equation that can be evidenced in other sources and one can observe the sequence of syntheses of these distinct concepts carried out at different moments of history.[49] This remark about the joint production of practices and bodies of knowledge also explains why restoring practices enables historians to make sense of clues.

For the sake of the analysis, one may thus distinguish between practices and the knowledge produced. However, in the last analysis, one can but observe their intimate relationship. More generally, in a sense, practices shaped to inquire into knowledge and carry out operations, far from being opposed to the results produced, can be considered as belonging to the knowledge produced in the framework of a knowledge activity.[50] These practices are not formed by spontaneous generation. They are, as much as the results, concepts and theories, the conscious products of normed activity. And knowledge about their ability to guide action and inquiry is transmitted along with the other outcomes of scientific activity. This conclusion accounts for their relative stability

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Lena Soler, *Introduction à l'épistémologie. Nouvelle édition revue et augmentée d'un chapitre* (Paris, 2009). The reader will find in this chapter, especially on pp. 300-302, a discussion of theses close to those which I present here. My analysis focuses here on the issue of clues.

[49] I sketch this thesis in K. Chemla, "De la synthèse comme moment dans l'histoire des mathématiques", *Diogenes* 160 (1992).

[50] In fact, like the concepts and results on which we concentrated in this chapter, ancient practices can be restored mainly through an examination of clues. I have not addressed the issue in this chapter to avoid overloading the argument. I refer the reader to my other publications on these topics.

and their collective dimensions. These features also explain why the description of practices can be of help in the interpretation of sources of the past.

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